

NOTE

WHAT ARE PROPOSITIONS 84 AND 85 OF EUCLID'S DATA ALL ABOUT?

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*C'est assez qu'on ait vu par là qu'il ne faut point/
Agir chacun de même sorte.*

La Fontaine, "L'Âne chargé
d'éponges, et l'âne chargé de sel"

Propositions 84 and 85 of Euclid's Data read:

PROPOSITION 84. *If two straight lines enclose a given area in a given angle, and one of them is greater than the other by a given line, then each of them will be given.*

PROPOSITION 85. *If two straight lines enclose a given area in a given angle and if their sum is given then each of them will be given.* [Euclid 1816, Vol. 3, 456]

Various authors, for example, Tannery [1882, 401 (read 84, 85 for 79, 80)], Heath [1921, Vol. 1, 423], and Van der Waerden [1954, 121], have supported their association of II, 5, 6 of the *Elements* with the theory of equations by a reference to Data 84, 85 [1]. Indeed if we let x and y be the lengths of the unknown sides whose sum $x + y$ is known to be a and if the area is b then these authors interpret Proposition 85 as representing in geometrical format, for the case of a right angle, the pair of equations

$$x + y = a; \quad xy = b.$$

In this note I wish to reexamine the whole question of a possible relationship between Propositions 84 and 85 and the equations.

First of all let us look at Proposition 85 more closely (84 is analogous) by working backward through the chain of propositions from the *Data* and theorems from the *Elements*.

From the *Data* itself, nineteen propositions are used according to Peyrard's indication. These in turn depend upon the following theorems from the *Elements*: I, 3, 8, 10, 11, 16, 23, 30, 31, 32, 34, 41, 43, 46; V, 16, 18, 22; VI, 1, 4, 6, 11, 13, 17, 18, 20, 22. Conspicuous by their absence are the results of

Book II and the "application of area" results I,44 and VI,28, 29 which also have been associated with equations; see, e.g., Van der Waerden [1954, 123].

As far as the *Data* itself is concerned, the key result is Proposition 58 which reads:

PROPOSITION 58. *If a given area is applied to a given line and if this area is deficient by a figure whose shape is given [2] then the lengths of the deficiency are given. [Euclid 1816, Vol. 3, 397]*

This statement is very reminiscent of VI,28 which shows how to apply a parallelogram, whose area is given, to a given line so that the applied parallelogram is deficient by a parallelogram similar to a given parallelogram. Further, the proofs of Proposition 58 and VI,28 both use I,43 (to prove a II,5 type result) and VI,18 (to construct a similar figure). The difference between these two results is one of aim or "raison d'être" and in modern terminology we might describe VI,28 as being the "constructive version of Proposition 58. In the same vein there is a "constructive" version of Proposition 85 which, as I shall now show, may be easily deduced from VI,28:

PROPOSITION 85'. *It is required to perform the following construction. We are given that the area of the parallelogram defined by two lines and their given included angle is equal to the area of a rectilinear region C (see Fig. 1) and that the sum of the two sides is equal to a line AB. We are to construct the two sides of the region [3].*

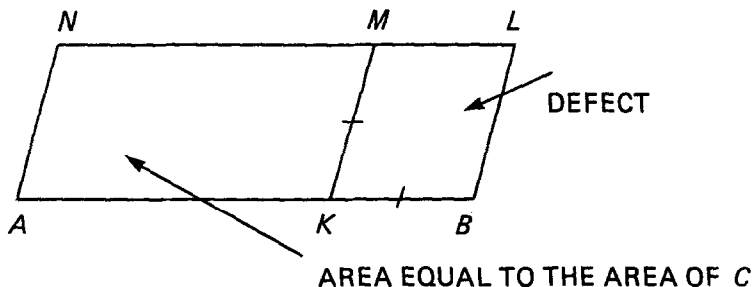


FIGURE 1

PROOF. Using VI,28, apply to line AB a parallelogram AKMN whose area equals the area of C and with the defect being a rhombus KBLM whose angle is the same as that of the given angle

between the lines. Then since $MK = KB$, the sum of the sides AK and KM of the parallelogram $AKMN$ is equal to AB . Thus the pair AK and KM is a solution of the problem.

If Euclid were interested in the geometric version of the equation set (1) then we could reasonably expect to see a statement close to that of Proposition 85' in the *Elements* or in the *Data*. Moreover, we would expect to find something like the above proof if he had wanted to indicate how the "application of areas," in particular VI, 28, was related to the solution of these equations.

If then Proposition 85' represents what the solution of equation set (1) looks like in geometric form, it may legitimately be asked what Proposition 85 represents? From the viewpoint of a modern mathematician one possible answer is that, whereas Proposition 85' represents a "constructive" proof that the set of equations (1) has a (unique) solution, Proposition 85 represents a nonconstructive "existence" proof. This would be analogous to the case of a pair of linear equations in two unknowns with a nonzero determinant for which one may give a constructive proof by actually solving the equations or an existence (and uniqueness) proof by arguing that the lines are not parallel and will therefore meet in (only) one point.

To give such an interpretation to Proposition 85 in the context of Greek mathematics is not, in my opinion, valid for it requires the a priori assumption that we are indeed dealing with equations here and that the Greeks did consider such things as the existence of solutions of equations [4].

I submit then that Propositions 84 and 85 did not, at the time of Euclid, have anything to do with the concept of "equations" but are simply two propositions from the *Data* which unfortunately have been singled out for special attention. The *Data* itself has been insufficiently investigated as a unit but the "plan" [Marinus 1947, 52] and the various statements of the propositions give the impression that the *Data* was meant to be a study of which geometrical quantities are "determined" when other geometrical quantities are "given." From this viewpoint, Propositions 84 and 85 do not present any unusual aspects.

As to the origins of the *Data* it is possible that the work may have arisen from philosophical considerations. Indeed the "Commentary" of Marinus, a student of Proclus who was active at the end of the fifth century, begins, "It is first of all necessary to determine what is datum," and this is followed by a discussion of various ancient and modern opinions on the matter [Marinus 1947, Chap. II; see also Chap. I, Sect. 2] Marinus criticizes Euclid [63] for only defining the concept of being "given" for area, ratio, lines, ..., instead of giving a general definition and he concludes that the best general definition of a "given" is something that can be constructed, obtained and reproduced.

While not wishing to assert a direct connection between the following three texts--two ancient and one medieval--and the *Data*, I mention them here because they may be at least related to the spirit behind the *Data*.

i. In *Meno* 86 E, Socrates speaks of geometers who when "asked, for example, 'Is it possible to inscribe a figure as a triangle in a given circle?' ... will reply 'I don't know yet whether it is possible. But I think I have a particular hypothesis that may be of use in this case'" [Thomas 1980, 52]; see [Bluck 1961, Appendix] for a recent summary of the various interpretations of this text.

ii. Speaking of Leon, a contemporary of Plato, in his "Catalogue of Geometers" Proclus says, "He also discovered diorismi, whose purpose is to determine when a problem under investigation is capable of solution and when it is not" [Proclus 1970, 55]; see Mugler [1958, 141] and Proclus [1948, 59] for discussions and examples of other uses of the word.

iii. In al-Biruni's *Chords* and in the trigonometrical portions of *Canon* we find both "constructive" and "existence" proofs of the same result. For example [al-Biruni 1910, 62; 1927, 6] the second "broken chord" theorem is used to prove that the side of the decagon is "known" if the diameter of the circumscribing circle is "known." But in addition we are shown how to actually calculate the side of the decagon in terms of the diameter. It is possible that the use of the broken chord results for "existence" proofs is ancient, for al-Biruni [1910, 13; 1927, 3] explicitly states that the first "broken chord" theorem is due to Archimedes.

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NOTES

1. Because of a referee's suggestion I refer, although originally I had planned not to do so, the reader to the recent dispute in the literature centered around Unguru [1975] which concerns the question of whether or not parts of Euclid are "algebra." In my opinion each part of Euclid must be read and understood as Greek mathematics without any a priori assumptions or association of results. As an example of this I mention the recent work by C. Taisbak [1982] whose explanation and presentation of book X will cause many readers to permanently reject Heath's version of the Euclidean text. The same referee is

correct when he points out that Heath [1921, Vol. I, 422] states that "The object of the proposition called a Datum is to prove that, if in a given figure certain parts or relations are given, other parts or relations are also given...." What I am contesting is Heath's statement, "Euclid shows how to solve [(1)] in Propositions 84, 85 ...," which hardly squares with his "It is to be observed that this form of proposition does not actually determine the thing or relation which is shown to be given, but merely proves that it can be determined...."

2. Definition 3 of the *Data* says: "Rectilinear figures, each of whose angles is given, and the ratios of whose interlying sides are given, are said to be given in shape" [Euclid 1816, Vol. 3, 302]. See also the discussion of "shape" given by Mugler [1958, 161].

3. The conclusion could have continued: "Furthermore we are to show that no other sides satisfy the requirements." I have left out the statement and proof from the text to emphasize that Proposition 85' is an immediate consequence of VI, 28. The uniqueness proof itself uses ideas from VI, 28 and with reference to the diagram accompanying VI, 28 in the Euclidean text proceeds as follows: Suppose that S is another division point of AB , say to the right of K . If we now consider the parallelogram $ASQT$ with $QS = SB$ then MQ will lie on the diagonal GB which in turn will imply that the gnomon defined by Q has a smaller area than the gnomon defined by M . But the area of each parallelogram is precisely the area of the corresponding gnomon, so that parallelogram $ASQT$ will have a smaller area than parallelogram $AKMN$. Thus the pair AS and SB is not a solution. In the same way if S is assumed to be to the left of K , then the area will be larger. Implicit in this proof is the assumption that both points K and S are to the right of the midpoint E . Indeed the maximum area occurs when S coincides with E , this being the message of VI, 27. There is, of course, a dual solution when AK is less than KM .

4. Slavutin [1979] proposes to give the meaning of the *Data* for the history of algebra and to show the relationship of the propositions to the theory of linear functions.

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