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$$p_{2n-1}^{(n-1)}(1) = \frac{(n-1)!}{2^{n-1}}C_{2n+1}. \tag{5.6}$$

For example,

$$p_5''(1) = 24 - 360 + 472 = 136,$$

in agreement with (5.6).

(*Please turn to page 465.*)

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HOW TO FIND THE "GOLDEN NUMBER" WITHOUT REALLY TRYING

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"*. . .I wish . . . to point out that the use of the golden section . . . has apparently burst out into a sudden and devastating disease which has shown no signs of stopping . . .*" [2, p. 521]

Most of the papers involving claims concerning the "golden number" deal with distinct items such as paintings, basing their assertions on measurements of these individual objects. As an example, we may cite the article by Hedian [13]. However measurements, no matter how accurate, cannot be used to reconstruct the original system of proportions used to design an object, for many systems may give rise to approximately the same set of numbers; see [6, 7] for an example of this. The only valid way of determining the system of proportions used by an artist is by means of documentation. A detailed investigation of three cases [8, 9, 10, 11] for which it had been claimed in the literature that the artist in question had used the "golden number" showed that these assertions were without any foundation whatsoever.

There is, however, another class of papers that seeks to convince the reader via statistical data applied to a whole class of related objects. The earliest examples of these are Zeising's morphological works, e.g., [17]. More recently we have Duckworth's book [5] on Vergil's *Aeneid* and a series of papers by Benjafield and his coauthors involving such things as interpersonal relationships (see e.g. [1], which gives a partial listing of some of these papers).

Mathematically we may approach the question in the following way. Suppose we have a certain length which is split into two parts, the larger being M and the smaller m. If the length is divided according to the golden section, then it does not matter which of the quantities, m/M or $M/(M + m)$, we use, for they are equal. But now suppose we have a collection of lengths and we are trying to determine statistically if the data are consistent with a partition according to the golden section. Authors invariably use $M/(M + m)$, but we may reasonably ask which of the two we should really use or whether or not it matters.

Our starting point is a remark by Dalzell in his review of Duckworth's book: "But Professor Duckworth always uses the more complex ratio $M/(M + m)$, which he describes as 'slightly more accurate.' Just the reverse is true. In the relatively few instances when the quotient is exactly .618 then $m/M = M/(M + m)$ and it does not matter which ratio is used. But in all other cases the more complex ratio is less sensitive to deviations from the perfect figure of .618" [4].

Let us designate m/M by x, then $M/(M + m)$ becomes $1/(1 + x)$. The golden number is $\Phi = (1 + \sqrt{5})/2$, and we let $\varphi = 1/\Phi$. We then have

Dalzell's Theorem: For all x in $[0, 1]$; $|1/(1 + x) - \varphi| \leq |x - \varphi|$.

Why should this result be true? Intuitively, we might reason that in writing $1/(1 + x)$ we are starting to form the continued fraction expansion of φ. We shall see later that in a sense our intuition is correct, but that there are limits to its validity. As a direct proof via continued fractions seems difficult, we use a roundabout approach.

Lemma: Let f be differentiable on $[a, b]$ with $|f'| \leq M$. If α is a root of $f(x) = x$ (i.e., a fixed point of f), then $|f(x) - \alpha| \leq M|x - \alpha|$ on $[a, b]$.

Proof: Mean value theorem.

Corollary: Dalzell's theorem.

Alternatively, we can obtain the estimate $|f(x) - f(y)| \leq [1/(1 + a)^2]|x - y|$ with $f(x) = 1/(1 + x)$, $0 \leq a \leq x, y$, by simple computation. This shows directly that f is a contraction operator with fixed point φ. In particular, when we restrict ourselves to an interval bounded away from 0, we see that the distortion caused by using $M/(M+m)$ instead of m/M is larger than that indicated by Dalzell's theorem.

Corollary: On the interval $[a, 1]$ where $a = \sqrt{2} - 1 = .414\ldots$,

$$\left|\frac{1}{(1 + x)} - \varphi\right| \leq \frac{1}{2}|x - \varphi|.$$

Note in fact that as x ranges from .5 to .75, $1/(1 + x)$ only ranges from .667 to .571.

Corollary: For x close to φ,

$$|1/(1 + x) - \varphi| \approx (\varphi^2)|x - \varphi| \qquad (\varphi^2 = 1 - \varphi = .381\ldots).$$

Because of its independent interest we now make a slight digression into continued fractions. We restrict ourselves to the unit interval and therefore write $[0, a_1, a_2, \ldots]$ for $1/(a_1 + 1/a_2 + \cdots)$.

Theorem: Let $\alpha \in [0, 1]$ have a periodic continued fraction expansion of the form $\alpha = [0, \overline{a_1, a_2, \ldots, a_k}]$. Then for any number x in $[0, 1]$,

$$|[0, a_1, a_2, \ldots, a_k, x] - \alpha| \leq |x - \alpha|.$$

Proof: Define f by $f(x) = [a_1, a_2, \ldots, a_k, x]$, then, by the periodicity, $f(\alpha) = \alpha$. Furthermore, $f(x) = \dfrac{Ax + B}{Cx + D}$, where the coefficients are integers which do not depend on x and satisfy $|AD - BC| = 1$ (A/C and B/D are, respectively, the $(k - 1)$st and kth convergents to α; see [12, Th. 175] and [15, Th. 7.3]). From this we obtain $|f'(x)| = 1/|(Cx + D)|^2 \leq 1$ on $[0, 1]$. The proof is concluded by use of the lemma.

Corollary: Dalzell's theorem.

Proof: $\varphi = [0, 1, 1, 1, \ldots]$; $[0, 1, x] = 1/(1 + x)$.

Remark: This theorem justifies our earlier intuitive remark, as to why Dalzell's theorem should hold; however, our intuition will lead us into difficulties unless we stop at the end of a period. Indeed, if $\alpha = [0, \overline{b_1, \ldots, b_k}]$ and $j < k$, then for $x = \alpha$, $x - \alpha$ is zero, whereas $[0, b_1, \ldots, b_j, x] - x$ is not zero.

Remark: The above approach can be used to place some results involving continued fractions in the domain of attraction of fixed points and contraction operators, but we shall not pursue this path here.

Remark: It is known that every periodic continued fraction is a quadratic surd, i.e., an irrational root of a quadratic equation with integral coefficients, and conversely ([10, Ths. 176, 177] and [15, Th. 7.19]). In the case of $\alpha = \varphi$, the corresponding equation is $x^2 + x = 1$ or $x = 1/(1 + x)$. One would thus be tempted to treat the general periodic case as follows: Suppose α satisfies $Ax^2 + Bx = C$. We rewrite this as $x = f(x) = C/(Ax + B)$, and would like to conclude that

$$|f(x) - \alpha| \leq |x - \alpha|$$

as above. However, we run into difficulty because we no longer have a control on f'.

Let us now turn our attention to the statistical aspects. We denote random variables by capital letters, expectation by E, variance by σ^2, and standard deviation by SD. We restrict ourselves to distributions with continuous densities concentrated on the unit interval. By the second corollary above (p. 407) and the Mean value theorem for integrals, we have immediately—

Theorem: If X is a random variable taking values in a small interval near φ, then the ratios $r_1 = |E(X) - \varphi|/|E(Y) - \varphi|$ and $r_2 = \mathrm{SD}(X)/\mathrm{SD}(Y)$ are both near Φ^2.

Now consider a general "aesthetic" situation involving lengths of various sizes. We should not be surprised that, rather than being controlled by some mystical numerical force, our ratios m/M occur randomly. Furthermore, in situations such as the lengths of sections in a poem, there will be a tendency to avoid the two extremes of complete asymmetry and equality, i.e., we can expect values relatively far away from 0 and bounded away from 1.

Thus we are led to consider the situation where X is uniformly distributed on a subinterval $[a, b]$ of the unit interval. In this case, $E(X) = (a + b)/2$ and $\sigma^2(X) = (b - a)^2/12$ [14, pp. 74, 101, 111] and straightforward calculations [14, p. 78] now show that the distribution functions of $Y = 1/(1 + X)$ assigns weight $(1/c - 1/d)/(b - a)$ to a subinterval $[c, d]$ of $[1/(1 + b), 1/(1 + a)]$. Furthermore,

$$E(Y) = \left(\frac{1}{b - a}\right) \cdot \ln\left(\frac{1 + b}{1 + a}\right) \quad \text{and} \quad \sigma^2(Y) = \frac{1}{(1 + a)(1 + b)} - [E(Y)]^2.$$

Note that if $[c, d]$ is contained in $[1/(1 + b), 1/(1 + a)]$ and also in $[a, b]$ then the distribution function of y assigns $1/cd$ times more weight to $[c, d]$ than does the unifrom distribution on $[a, b]$. Under these conditions, if $[c, d]$ is a small subinterval about $1/\Phi$, then this ratio is approximately $\Phi^2 = 2.618$, i.e., for a large sample over two and one-half times as many values of the transformed data as of the untransformed values will lie in the interval. Also note that the weight assigned by the distribution of Y to an interval $[c, d]$ depends only on the length of the interval $[a, b]$ and not on the actual values of the endpoints.

In fact, numerical computation shows that even for large intervals relatively far away from 0 and bounded away from 1 the ratios r_1 and r_2 as well as the probability ratios will not be too far from 2.6. To illustrate this situation, let us suppose that our ratios are uniformly distributed on [.45, .70] so that the average value is .575 and the standard deviation .072. For a large sample, only 16% of the values will fall in the subinterval [.60, .64]. If we now transform the data, the mean is .636 and the standard deviation only .029. This means that for a sample size of 20 or so it is almost sure that the mean will lie in the interval [.607, .665]. Furthermore, for a large sample, 42% of the actual values of $1/(1+x)$ will lie in our subinterval [.60, .64]. If we look at [.59, .65], then the probabilities are 24% and 62%.

Finally, to support our claim that the various seemingly impressive results in the literature are really due to an invalid transformation of data from a more or less uniform distribution, we mention two case studies.

The first is due to Shiffman and Bobko [16] who considered linear portionings and concluded that a uniform distribution of preferences was indeed the most likely hypothesis.

The other, a study on Duckworth's data, was done by the present author in connection with a historical study [3] of the numerical treatment of Φ by Hero of Alexandria who lived soon after Vergil. If we consider the first hundred entries in Duckworth's Table I, then the range of the m/M values is from $4/7 = .571$ (four times) to $2/3 = .667$ (twelve times). If this range is split up into five equal parts, then the five subintervals contain 10, 25, 33, 15, and 17 values, respectively. When we look at the actual values, we note that the Fibonacci ratios 3/5, 5/8, and 13/21 appear 15, 16, and 2 times, respectively. In other words, 2/3 of the ratios are *not* Fibonacci approximations to the "golden number." If we compute means and standard deviations, then for the m/M ratios we obtain the values .621 and .025 as opposed to the values .616 and .010 for the $M/(M + m)$ ratios, which only range from .600 to .637. It is interesting to note that if Vergil had used the end values 4/7 and 2/3 fifty times each, then the average would have been

$$\frac{1}{2}\left(\frac{4}{7} + \frac{2}{3}\right) = \frac{13}{21},$$

which is a good Fibonacci approximation to φ. This only proves once more how deceiving averages can be. A similar study of the sixteen values in Duckworth's Table IV—the main divisions—reveals that not a single Fibonacci ratio appears. The m/M values range from .594 to .663 with a mean of .625 and standard deviation of .021 as opposed to values of .615 and .008 for the $M/(M + m)$ values.

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EXTENDED BINET FORMS FOR GENERALIZED QUATERNIONS OF HIGHER ORDER

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In a prior article [4], the concept of a higher-order quaternion was established and some identities for these quaternions were then obtained. In this paper we introduce a "Binet form" for generalized quaternions and then proceed to develop expressions for extended Binet forms for generalized quaternions of higher order. The extended Binet formulas make possible an approach for generating results which differs from that used in [4].

We recall from Horadam [1] the Binet form for the sequence $W_n(a, b; p, q)$, viz.,

$$W_n = A\alpha^n - B\beta^n$$

where

$$W_0 = a, \qquad W_1 = b$$

$$A = \frac{b - a\beta}{\alpha - \beta}, \quad B = \frac{b - a\alpha}{\alpha - \beta}$$

and where α and β are the roots of the quadratic equation

$$x^2 - px + q = 0.$$

We define the vectors $\underline{\alpha}$ and $\underline{\beta}$ such that

$$\underline{\alpha} = 1 + i\alpha + j\alpha^2 + k\alpha^3 \quad \text{and} \quad \underline{\beta} = 1 + i\beta + j\beta^2 + k\beta^3,$$