A "Very Pleasant Theorem"

Roger Herz-Fischler

Roger Herz-Fischler teaches mathematics and culture at Carleton University. After teaching a course in mathematics for students of architecture he realized that some statements made to students about the "golden number" were completely without an historical basis. This has led to a "career change" and various scholarly articles; e.g., the recent "Golden Numberism in France from 1896 to 1927." The book *A Mathematical History of Division in Extreme and Mean Ratio* is being followed by *The Golden Number: A Philosophical, Historical, Sociological and Analytic Study*. The sociological aspect will examine the spread of the golden number myth in various mathematically literate and illiterate circles.

The independent discovery of a result is common throughout the history of mathematics, particularly in geometry. This article is about a simple result concerning certain right triangles that has been repeatedly rediscovered in a variety of contexts and it discusses how the result has been rephrased, extended and reproved in some not so simple ways. We will even see how it leads to an approximate fold-up paper model of the Great Pyramid of Egypt!

The basic form of the result—which has been given a name based on its description by Kepler—is, in modern notation, the following:

A "very pleasant theorem." *If a right triangle has its sides in geometric ratio* $1: \sqrt{G} : G$, *then* $G = (1 + \sqrt{5})/2$.

An analytic proof is immediate from the Pythagorean theorem which gives:

$$1 + G = G^2, \tag{1}$$

whose positive solution is $G = (1 + \sqrt{5})/2$. The reader will recognize this latter number as being the "golden number" and thus should not be surprised to learn that among the rediscoverers of the result were various "golden numberists." A right triangle having the ratio $1: \sqrt{G} : G$ among its sides will be referred to as a Kepler triangle.

Magirus and Kepler (1597). The first written statement of the result appears in a letter that the mathematician–astronomer Kepler wrote to his former professor Michael Mästlin; see [**4**, p. 159]. Kepler states that part of the result is due to a music professor named Magirus, but it is not clear what the latter should be credited with. Kepler writes, "I...show my thankfulness to [Magirus] because of this, that as by his very pleasant theorem he has pleased me among others with a new enthusiasm for geometry...I have changed it into another form such that I think I will easily persuade even Magirus himself to think that it is entirely mine."

To see how Kepler extended the basic result, we start with a Kepler triangle *EFD* (Figure 1) and then determine point A on the extension of EF by taking AD perpendicular to ED. Theorem VI,8 of Euclid's *Elements* states that when the perpendicular DF is constructed the three triangles EFD, EDA and DFA are similar and this permits us to make the following statements:

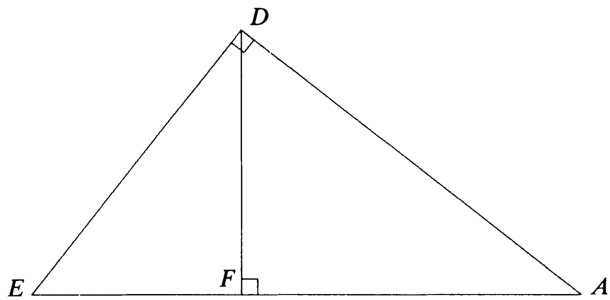

Figure 1

1. $AF = ED$

Proof. Since the sides of a Kepler triangle EFD are, by definition, proportional, $ED : DF = DF : FE$. But by similar triangles $AF : DF = DF : FE$ and for both proportions to hold we must have $AF = ED$.

2. $AE : AF = AF : FE$, i.e., the ratio of the line AE to the larger segment AF is the same as the ratio of AF to the smaller segment FE. In the technical language of Euclid's *Elements* VI, def. 3, point F divides AE in extreme and mean ratio; in numerical terms the common ratio is G.

Proof. By similar triangles $AE : ED = ED : FE$. Substitution of AF for ED (from (1)) gives the result.

These statements, and essentially the same proofs, are what we find in the 1597 letter, except that Kepler started with the assumption that line AE is divided in extreme and mean ratio and showed that the sides of the larger triangle are proportional. The relationships among the various sides and segments can be expressed in numerical terms by starting with $FE = 1$ and then using the definition of the Kepler triangle and the above results (Figure 2). Note that the relationship $FE + AF = AE$ is the geometric equivalent of (1). The reader can find other relationships involving G from the diagram.

Before turning to the rediscoverers of the theorem it should be noted that Kepler's letters were not published until 1857–1871, so that it is hardly possible that this was the source of the rediscovery.

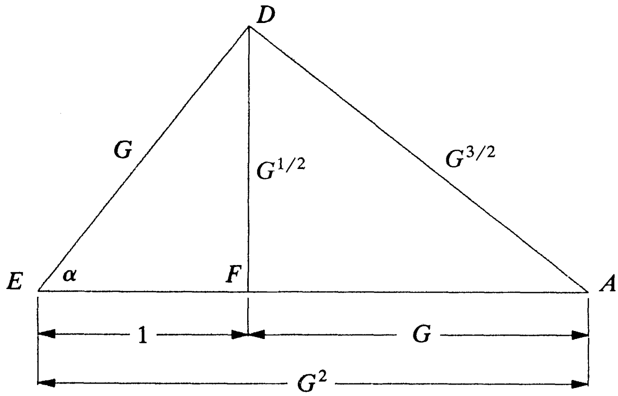

Figure 2

Mascheroni (1797). Mascheroni is known for his *Geometry of the Compass* which deals with geometric constructions involving only a compass with a variable opening. A construction of the Kepler triangle is found in Section 180 of the 10th part. Mascheroni's method is to suppose that the hypotenuse AE (Figure 3) is the diameter and that we have divided AE in extreme and mean ratio at F with AF being the larger segment (a problem which Mascheroni had considered earlier). Now at E we swing an arc of radius AF which meets the semicircle on diameter AE at D. Since triangle ADE is a right triangle and $ED = AF$ we have precisely the situation of Figure 1 and of Kepler's result.

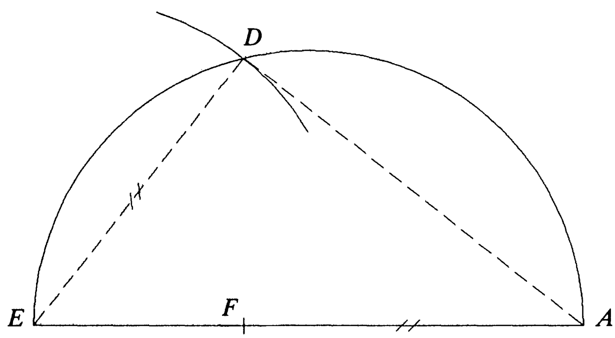

Figure 3

Wiegand (1847). The Kepler triangle also appeared, but in a partially algebraicized version, in an 1847 German geometry book written by Wiegand. This suggests that the result may have been fairly widely known at the time.

Röber (1855). This virtually unknown author is in fact one of the co-founders of "golden numberism." Röber's aim was to convince the reader, via quantities obtained from the Kepler triangle, that the golden number was used to design virtually all the Egyptian pyramids *except* the Great Pyramid. For the Great Pyramid Röber states, in accordance with previous authors, that the base to height ratio was 8:5. For details of the 8:5 theory, as distinct from a golden number theory, as well as a description of Röber's very involved use of the quantity G, see [**2**, **5**]).

In Figure 4, which is a modification of the diagram in Röber's *The Egyptian Pyramids*, we take $AF = FC = 1$ and $BF = 1/2$. Using BC as a radius, we obtain point E on the extension of AF with $BE = BC = \sqrt{5}/2$ and this in turn implies that $AE = (1+\sqrt{5})/2 = G$. When we draw the semicircle ADE and the perpendicular DF, we have the three Kepler triangles as in Figure 1.

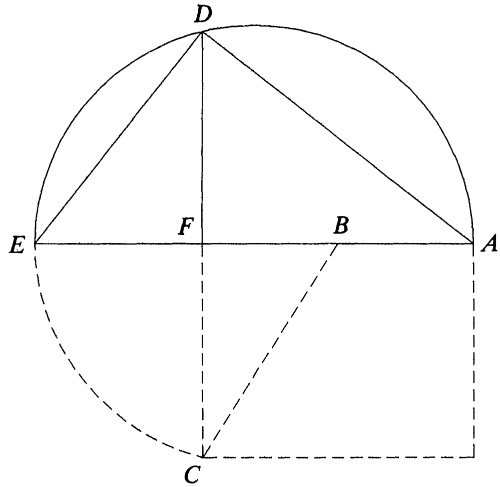

Figure 4

This explanation of the diagram is not however the development presented by Röber who takes the following roundabout route. The dotted-line construction is essentially that of *Elements* II,11 and from the proof of this result we learn that FE is the larger segment when AF is divided in extreme and mean ratio. Theorem XIII,5 of the *Elements* now tells us that the line AE is also divided in extreme and mean ratio with AF being the larger segment. Next height DF is taken to be the mean proportional between AF and FE and by the converse of VI,8 triangle ADE is a right triangle which in turn implies that arc ADE is a semicircle. We are thus back to the situation of Figure 1 with three Kepler triangles.

So far our discussion has involved only plane geometry, but now however the story becomes three-dimensional!

Ballard (1882). Ballard was an English railway engineer who saw the pyramids on his way to Australia and this inspired his book, *The Solution of the Pyramid Problem or, Pyramid Discoveries with a New Theory as to Their Ancient Use* [**1**]. Based on various measurements by others, Ballard had concluded that for the Great Pyramid the ratio of the slant-height of a triangular face to the half-base was 34:21. Since this ratio is an approximation to G by means of Fibonacci numbers, Ballard, who was well aware of the approximation, is essentially stating that the cross-sectional triangle of the Great Pyramid is made up of two Kepler triangles (Figure 5).

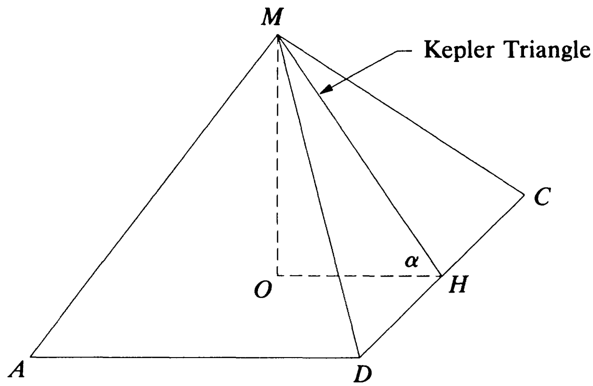

Figure 5

Ballard gives a historical justification for his theory by invoking a statement supposedly due to the Greek historian Herodotus. This statement, which has a very interesting history in its own right (see [**3**] for details and the summaries in [**7**] and [**8**]), says that:

> The Great Pyramid was built so that the area of each face has the same area as a square whose side equals the height.

The reader can easily verify (see the above articles) that this statement implies that the secant of the base angle of the cross-sectional triangle is G, and this is precisely the case for the Kepler triangle (Figure 2).

Ballard then continues the development of his theory by the construction of the "Star Cheops" which "...is the geometric emblem of extreme and mean ratio and the symbol of the Egyptian Pyramid Cheops." The Cheops referred to is the Pharaoh who built the Great Pyramid.

Ballard's construction starts with a circle of radius 1, with center at 0, and the inscribed pentagram (Figure 6). By using *Elements* XIII,8 and similar pentagons it

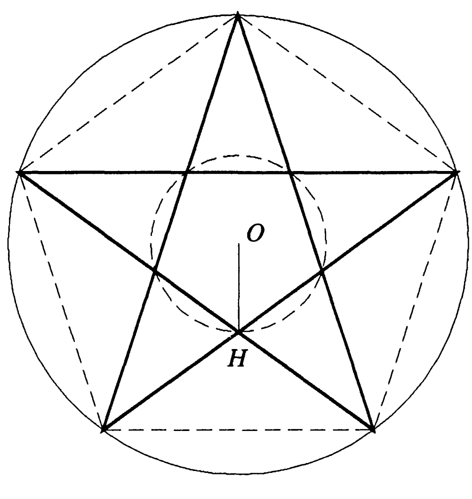

Figure 6

can be shown that OH, the radius of the circumscribed circle of the small pentagon, is equal to $(1/G)^2$.

About this latter circle we circumscribe (Figure 7) a square $ABCD$. This square, which corresponds to the base of the pyramid, has a half-side equal to the radius of the circle, i.e. $OH = (1/G)^2$. Now from the four sides of the base $ABCD$ we draw the four triangles which represent the faces of the pyramid, finding M as the point where OH meets the unit circle. The slant-height of the triangle CMD is the length $HM = OM - OH = 1 - (1/G)^2$ and if we divide both sides of (1) by G^2 we find that $HM = (1/G)$.

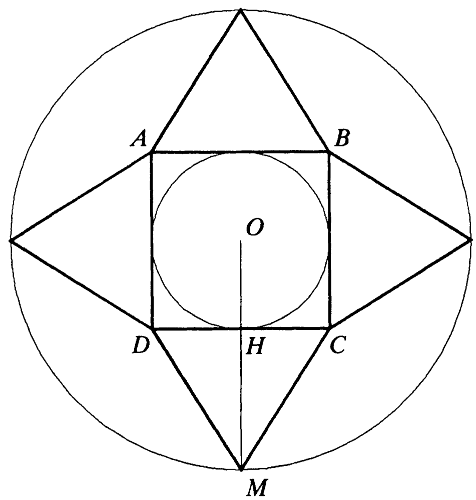

Figure 7
Star Cheops.

It should be noted that this last statement is equivalent to saying that HM is obtained by dividing the radius OM in extreme and mean ratio. This remark simplifies the construction, but Ballard started with the pentagram for symbolic reasons.

We can now obtain a pyramid by folding up the triangular sides of this "Star Cheops." If we consider the half cross-sectional triangle OHM of this fold-up pyramid then the secant of the base angle is $HM/OH = G$. Since G, as stated above, is also the secant of the base angle of the Kepler triangle, the shape of the our folded paper pyramid will be the same as the pyramid obtained from the Kepler triangle (Figure 5) and this pyramid in turn has a shape very close to that of the Great Pyramid itself.

Epilogue. A referee kindly drew my attention to the very interesting 1990 article [**6**] which considers the intersection points of the graphs of four pairs of trigonometric and hyperbolic functions. It turns out that $\sqrt{G}$ is a common value for the ordinate in the four cases and the article examines why this is so.

Consider the equation $\csc x = \tan x$. If we think of x as being an angle of a right triangle, then we have hypotenuse : opposite = opposite : adjacent, i.e. the sides are in a geometric ratio. The Pythagorean theorem shows that the common ratio is $\sqrt{G}$ so that once again we have the Kepler triangle. Thus the problem of the intersection points of the graphs of trigonometric functions has led to still another approach to our result. Furthermore in the introduction to the article the authors refer to the approach which uses the Kepler triangle as being a "pleasant explanation"!

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How to Dissect an Ankle

Nora said. "But this is just a theory, isn't it?"

"Call it any name you like. It's good enough for me.

"But I thought everybody was supposed to be considered innocent until they were proved guilty and if there was any reasonable doubt, they—"

...

"When the murders are committed by mathematicians," I said, "you can solve them by mathematics. Most of them aren't and this one wasn't. I don't want to go against your idea of what's right and wrong, but when I say he probably dissected the body so he could carry it into town in bags I'm only saying what seems most probable."

Dashiell Hammett, *The Thin Man*, Vintage Books, NY, 1933, p. 195.